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گۆڤاری زانكۆی سلیمانی- بەشی A

On centers for Generalized Abel Differential Equation

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Abstract:

A new condition is given for generalized Abel differential equation to have a center. We apply the results to some polynomial differential systems in the plane to find necessary and sufficient center conditions.

Keywords: Abel differential equation, center condition, planar polynomial vector field.

1. Introduction:

We consider generalized Abel differential equation of the form

$$\frac{dx}{dt} = \sum_{k=2}^{n} p_k(t)\ x^k \qquad \textbf{(1)}$$

where p_k(t) are real valued continuous functions on [0, *ω*] and $x \in R$. For a fixed real number ω we seek information about the number of solutions of (1) satisfying

x(0) = x(ω); we say that such solutions are periodic of period *ω*. The solution *x = 0* is called a center for (1) if all solutions starting in a neighborhood of *x = 0* are ω-periodic solutions. Furthermore, if they are isolated are called limit cycles of (1). Let *x* (t, c) be the solution of (1) such that *x* (0, c) = *c*.

We write, for $0 \leq t \leq \omega$ and c in a neighborhood of the origin $x(t;0,c) = \sum_{i=1}^{\infty} a_i(t)\ x^i$.

Since $x(0;0,\mathrm{c}) = \mathrm{c}$, we must have $a_1(0) = 1$ and $a_i(0) = 0$ for $i \geq 2$. The values of $a_i(\omega)$ is called the *i* th focal value. The origin is a center if and only if $a_1(\omega) = 1$ and $a_i(\omega) = 0$ for $i \geq 2$ (see[2,3]).

Let b_k denote the operator multiply by p_k and integrate. For example

$$b_2 b_4 b_3(\mathrm{t}) = \int_0^t p_2(s) \int_0^s p_4(r) \int_0^r p_3(u)\ du\, dr\, ds.$$

Abel equations arise in several circumstances but perhaps the main reason for their recent study is connected to the family of systems

$$x^{\bullet} = y + M(x,y)$$
$$y^{\bullet} = -x + N(x,y) \qquad \textbf{(2)}$$

when *M* and *N* are homogenous polynomials of the same degree *n≥2*. For systems of the form (2), the origin is a singular point of focus type fine (if it is a center for the correspo-nding linearzed system). Limit cycles bifurcate out of a fine focus when the coefficients of M and N are perturbed suitably; limit cycles so generated are said to be of small amplitude. A transformation due to Cherkas allows us to bring these systems to the form (1) where p_i are now trigonometric polynomials. There are also transformations to Abel-type equations for more general systems; see [3]. This trigonometric Abel equation has been used in a large number of works in order to estimate the number of limit cycles or obtain center conditions.

2. Preliminaries:

We shall give the two main results on which this work is based. The first is a result due to Devlin [4], which concern center conditions for (1).

Theorem 1[4]. Suppose that there is a differentiable function $\sigma(t)$ such that $\sigma(0)=\sigma(\omega)$ and that for each $k\in\{2,3,\dots,n\}$, there is a continuous functions f_k defined on $I=\sigma([0,\omega])$ such that $p_k(t)=f_k(\sigma(t))\,\sigma^{\bullet}(t)$. Then the origin is a center for the differential equation (1).

Also the second result due to Devlin [4], which enables to give the formula of $a_n(t)$ to find center conditions of (1).

Theorem 2[4]. For *$n\neq 1$,*

$$a_n=\sum_{i=2}^{n}(\mathrm{n}-\mathrm{i}+1)a_{\mathrm{n-i+1}}\,b_i\,. \qquad \textbf{(3)}$$

3. Center of Abel equations:

In this section, first we give a general condition for a center. For this purpose we must prove the following results in below.

Theorem 3. Suppose that p_j has mean value zero and for $k\in\{2,3,\dots,n\}\backslash\{j\}$, there is a continuous functions f_k defined on $I=\sigma([0,\omega])$ such that $p_k(t)=f_k(\sigma(t))\,\sigma^{\bullet}(t)$ where $\sigma(t)=\int_0^t p_j(s)\,ds$. Then the origin is a center for the differential equation (1).

Proof. Suppose that $\int_0^{\omega}p_j(s)\,\mathrm{ds}=0$ and $p_k=f_k(\sigma)\,\sigma^{\bullet}$ where $\sigma(t)=\int_0^{\mathrm{t}}p_j(s)\,\mathrm{ds}$ for $k\in\{2,3,\dots,n\}\backslash\{j\}$.Then $\sigma(0)=\sigma(\omega)$ and $p_k=f_k(\sigma)\,\sigma^{\bullet}$ for $k=2,3,\dots,n$. Hence the hypothesis of Theorem 1 are satisfied, then the origin is a center for (1).

Remark1. Theorem 3 includes known conditions for the origin to be a center: $p_k=0$ for $k=\{2,3,\dots,n\}\backslash\{j\}$ and $\int_0^{\omega}p_j(s)\,\mathrm{ds}=0$ (see corollary 7.3 in [4]). As a consequence of Theorem 3, one has obtained the following result.

Corollary 1. If $p_k=c_k\,p_j$ and p_j has mean value zero where c_k is a constant for $k=\{2,3,\dots,k\}\backslash\{j\}$. Then the origin is a center for the differential equation (1).

In this paper we consider the following natural extension of the Abel equation $x^{\bullet}=p_n(t)\,x^n+p_m(t)\,x^m$ with n > m >1. **(4)**

Lemma 1. Assume that the differential equation (4) has a center at the origin. Then $b_n(\omega)=b_m(\omega)=0$ and $(n+j-m+1)\quad a_{n+j-m+1}\,b_m(\omega)+a_{j+1}b_n(\omega)=0$ for j= 0,1,2,….

Proof. Suppose that the origin is a center for (4), then $a_1(\omega)=1$ and $a_i(\omega)=0$ for *i≥2.* Since $b_{m-i}(\omega)=0$ for $i\geq 1$ and $a_s\equiv 0$ for $\mathrm{s}\leq 0$, hence from Theorem 2, we obtain that $a_k\equiv 0$ for $\mathrm{k}\leq\mathrm{m}-1$ and $a_m=a_1b_m+(\mathrm{m}-\mathrm{n}+1)a_{\mathrm{m-n+1}}b_n$. Since $a_{\mathrm{m-n+1}}\equiv 0$ then $b_m(\omega)=0$. By equation (3), $a_n=(\mathrm{n}-\mathrm{m}+1)a_{\mathrm{n-m+1}}b_m+a_1b_n$. Since $a_l=0$ or $a_l=\mathrm{c}\,{b_m}^{r}$ for $m\leq l\leq\mathrm{n}$ where c is a constant and $\mathrm{r}\in\mathrm{N}$, then $b_n(\omega)=0$. Finally from (3), we obtain that

$$a_{n+j}(\omega)=(n+j-m+1)\,a_{n+j-m+1}\,b_m(\omega)+a_{j+1}b_n(\omega)=0$$

for j= 0,1,2,… as we wanted to prove.

Remark2.

1. Notice that for the case m = 1, the equation (4) is a Bernoulli's equation which can be explicitly solved. In this case it is easy to see that equation (4) has a center at the origin if and only if $b_n(\omega)=b_1(\omega)=0$.
2. From Lemma 1 the origin is not a center for (4) if either $b_m(\omega)\neq 0$ or $b_n(\omega)\neq 0$.

4. Applications to rigid polynomial planar systems:

In this section we apply the results of section 3 for studying the most important problem in the qualitative theory of planar vector fields is the study of center focus problem.

Let us start giving our motivation to study the planar systems whose angular speed is constant are usually called rigid systems. When the origin is a non degenerate they can be written as

$$x^{\bullet} = y + x\,F(x,y)$$
$$y^{\bullet} = -x + y\,F(x,y) \qquad \textbf{(5)}$$

where *F(x,y)* is an arbitrary function of variables x and y. For these systems the center focus problem is equivalent to the isochronicity problem. This is one of the reasons for which they have already been studied by several authors; see for instance [1,5]. Our results are applicable to different systems as it is illustrated by the following corollary.

Corollary 2. The origin of the family of rigid planar vector fields (5) is a center where $F(x,y) = (x+y)(d_1 + (x-y)\sum_{i=2}^{n} d_i (x+y)^{i-2})$, $d_i \in R$.

Proof. In this case, the system in polar coordinates becomes

$$r^{\bullet} = d_1(cos\theta + sin\theta)\,r^2 + \sum_{i=2}^{n} d_i(cos\theta - sin\theta)(cos\theta + sin\theta)^{i-1}\,r^{i+1}$$

$$\theta^{\bullet} = 1$$

By taking r as a function of θ, we get the single differential equation

$$\frac{dr}{d\theta} = d_1(cos\theta + sin\theta)\,r^2 + \sum_{i=2}^{n} d_i(cos\theta - sin\theta)(cos\theta + sin\theta)^{i}\,r^{i} = \sum_{k=2}^{n} p_k(\theta)\ r^k$$

which is a differential equation of type(1). Let $\sigma(\theta) = \int_0^{\theta}(cos\theta + sin\theta)d\theta$, then $\sigma(2\pi) = 0$ and $p_k(\theta) = d_i\sigma^k$ for $k =$ *3,4…,n.* By Theorem 3 all solutions in a neighborhood of $r = 0$ are 2π periodic and hence the origin is a center for the corresponding polynomial system.

Using the same technique of proof in Corollary 2, the following result holds**.**

Corollary 3. The origin of the family of rigid planar vector fields (5) is a center where $F(x,y) = (y-x)(d_1 + (x+y)\sum_{i=2}^{n} d_i (y-x)^{i-2})$, $d_i \in R$.

Remark3. Since the rigid systems in Corollaries 2 and 3 has a unique singular point at the origin, then there is no limit cycle.

2. It was shown in [5] that the system

$$x^{\bullet} = y + x\,(a + f_n(x,y))$$
$$y^{\bullet} = -x + y\,(a + f_n(x,y))$$

has no limit cycles and the origin is a center if $a = B = 0$ and there are no periodic solutions if $a^2 + B^2 \neq 0$ and $a\,B \geq$, where a is a real parameter, $f_n(x,y)$ is a homogeneous polynomial of degree *n* and $B = \int_0^{2\pi} f_n(cos\theta + sin\theta)d\theta$.

In this paper we also discuss polynomial systems of the form

$$x^{\bullet} = -y + x\,(f_2(x,y) + f_n(x,y))$$
$$y^{\bullet} = x + y\,(f_2(x,y) + f_n(x,y)) \qquad \textbf{(6)}$$

where $f_n(x,y)$ is a homogeneous polynomial of degree n. Sufficient and necessary conditions for the origin to be a center are obtained in [6,7] for *n =2* and 3 and in [8,9] for n = 4 and 5. In general it is very difficult to find sufficient and necessary conditions for the origin to be a center for system (6) due to increasing expansion of computation during the management of large expressions. Hence we study special case of system (6).

Our main result in this section is the following.

Theorem 4. The origin is a center for system

$$x^{\bullet} = -y + x\,(a\,x^2 + bx\,y + c\,y^2 + \mathrm{d}\,x^{n-1} + e\,x^{n-2}\,y + f\,x^{n-3}y^2) \quad \textbf{(7)}$$

$$y^{\bullet} = x + y\,(a\,x^2 + bx\,y + c\,y^2 + \mathrm{d}\,x^{n-1} + e\,x^{n-2}\,y + f\,x^{n-3}y^2)$$

if and only if the following conditions are satisfied:

$$\mathrm{a + c = f + (n-2)d = a\,e - b\,d} = \mathrm{a}\,d^2 = 0 \quad \textbf{(8)}$$

where n is odd number and $n > 3$.

Proof. The system (7) in polar coordinates r and θ now takes the form

$$\frac{dr}{d\theta} = \mathrm{p_3}(\theta)r^3 + \mathrm{p_n}(\theta)r^n \quad \textbf{(9)}$$

where $\mathrm{p_3}(\theta) = a\,cos^2\theta + b\,cos\theta\,sin\theta + c\,sin\theta^2\theta$ and $\mathrm{p_n}(\theta) = d\,cos^{n-1}\theta + e\,cos^{n-2}\theta\,sin\theta + f\,cos^{n-3}\theta\,sin\theta^2\theta.$

First we proceed to prove the sufficiency of the conditions given in this Theorem. From equations (8) and after simple calculations, we have that either:

(i) a = c = d = f = 0 or

$(ii)\; a = b = c = 0, f = (2-n)d = 0$ or

$(iii)\; d = e = f = 0, c = -a.$

Case (i) is straightforward, the system is

$$x^{\bullet} = -y + x\,(bx\,y + e\,x^{n-2}\,y)$$

$$y^{\bullet} = x + y\,(bx\,y + e\,x^{n-2}\,y)$$

which is symmetric in the *y*-axis. So the origin is a center. When condition (ii) of Theorem holds, the equation (9) becomes

$$\frac{dr}{d\theta} = (d\,cos^{n-1}\theta + e\,cos^{n-2}\theta\,sin\theta + (2-n)d\,cos^{n-3}\theta\,sin\theta^2\theta)\,r^n.$$

Since

$$\int_0^{2\pi}(d\,cos^{n-1}\theta + e\,cos^{n-2}\theta\,sin\theta + (2-n)d\,cos^{n-3}\theta\,sin\theta^2\theta)d\theta = 0.$$

By Theorem 3 all solutions in a neighborhood of r = 0 are 2π periodic and hence the origin is a center for the corresponding system. In case (iii) the equation (9) becomes

$$\frac{dr}{d\theta} = a\,(cos^2\theta - sin\theta^2\theta) + b\,cos\theta\,sin\theta$$

Since $\int_0^{2\pi}(a\,(cos^2\theta - sin^2\theta) + b\,cos\theta\,sin\theta)d\theta = 0$, the origin is a center by Theorem 3. Therefore, the sufficient part of Theorem is proved.

To prove the necessity of the conditions is derived from the focal values by computer algebra system with Maple. From Lemma 1 if the origin is a center for equation (9) then must be

$$a_3(2\pi) = b_3(2\pi) = 0 \quad , \quad a_5(2\pi) = 3{b_3}^2(2\pi) = 0, a_n(2\pi) = b_n(2\pi) = 0$$

$$a_{n+2}(2\pi) = (n\,b_n b_3 + 3\,b_3 b_n)(2\pi) = 0$$

$$a_{n+4}(2\pi) = \Big((n+2)(n\,b_n b_3 + 3\,b_3 b_n)b_3 + 5{b_3}^2\,b_n\Big)(2\pi) = 0 \quad \textbf{(10)}$$

After simple calculation we get $b_3(2\pi) = \int_0^{2\pi} p_3(\theta)d\theta = \pi\,(a + c)$.

Since $\int_0^{2\pi} cos^n\,sin^m\theta\,d\theta = \begin{cases} 0 & if\ n\ or\ m\ is\ odd \\ 2\pi\frac{(n-)!!(m-)!!}{(n+m)!!} & if\ n\ and\ m\ are\ even \end{cases}$

then $b_n(2\pi) = \int_0^{2\pi} p_n(\theta)d\theta = 2\pi\frac{(n-4)!!}{(n-1)!!}\,((n-2)\,d + f)$ if n is odd.

From equations (10), we must be $c = -a$ and $f = (2-n)d$ if the origin is a center of system (7). Substituting $c = -a$ into $p_3(\theta)$, then ${b_3}^k(2\pi) = 0$ for k=1,2,3,…, hence $a_5(2\pi) = \cdots = a_{n-1}(2\pi) = 0$. Now Substituting $c = -a$ and $f = (2-n)d$ into $p_3(\theta)$ and $p_n(\theta)$, then after tedious computation $a_{n+2}(2\pi) = 2\pi(2n-6)\frac{(n-2)!!}{(n+1)!!}\,(a\,e - b\,d)$.

Also if the origin is a center for system(7) then $a\,e - b\,d = 0$ **(11)**

Now if $a = 0$.From equation (11), we have that either (i) $b = 0$ or (ii) $d = 0$. For case (i) and (ii), then conditions (8) are obtained. If $a \neq 0$, substituting $e = \frac{b\,d}{a}$ into $p_n(\theta)$, then $a_{n+4}(2\pi) = -2\pi(n-3)(n^2 + 2\,n + 5)\frac{(n-2)!!}{(n+3)!!}\,a^2\,d$. If the origin is a center then $a^2\,d = 0$. Thus we can obtain the relations (8). This complete the proof of the Theorem.

To find the maximum number of small amplitude limit cycles which can bifurcate from the origin. From our calculations in Theorem 4, if $\mathrm{a} + \mathrm{c} = 0$, $\mathrm{f} + (\mathrm{n}-2)\mathrm{d} = 0$ and $\mathrm{a\,e} - \mathrm{b\,d} = 0$ then $a_3(2\pi) = a_5(2\pi) = \cdots = a_n(2\pi) = a_{n+2}(2\pi) = 0$ and $a_{n+4}(2\pi) = -2\pi(n-3)(n^2 + 2\,n + 5)\frac{(n-2)!!}{(n+3)!!}\,a^2\,d$, which is different from zero if a and d are different from zero, and therefore we obtain a fine focus of order $\frac{n+3}{2}$ for system (7). Therefore we have the following result.

Corollary 4. The maximum number of small amplitude limit cycles which can bifurcate from the origin is at least $\frac{n+3}{2}$ for system (7).

Conclusion:

In this work we proved new sufficient conditions for generalized Abel differential equation to have a center. The results is applied to found necessary and sufficient center conditions for the Poincare's system. Also found the maximum number of small amplitude limit cycles which can bifurcate from the origin of rigid system.

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